SIMULTANEOUS LINEAR INEQUALITIES: YESTERDAY AND TODAY

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Abstract. This is an short overview of the recent tendencies in the theory of linear inequalities that are evoked by Boolean valued analysis.

1. Agenda

Linear inequality implies linearity and order. When combined, the two produce an ordered vector space. Each linear inequality in the simplest environment of the sort is some half-space. Simultaneity implies many instances and so leads to the intersections of half-spaces. These yield polyhedra as well as arbitrary convex sets, identifying the theory of linear inequalities with convexity.

Convexity reigns in the federation of geometry, optimization, and functional analysis. Convexity feeds generation, separation, calculus, and approximation. Generation appears as duality; separation, as optimality; calculus, as representation; and approximation, as stability (cp. [1]).

This talk addresses the origin and the state of the art of the relevant areas with a particular emphasis on the Farkas Lemma (cp. [2]). Our aim is to demonstrate how Boolean valued analysis may be applied to simultaneous linear inequalities with operators.

This particular theme is another illustration of the deep and powerful technique of "stratified validity" which is characteristic of Boolean valued analysis.

2. Founding Fathers

Linearity, inequality, and convexity stem from the remote ages (cp. [3]–[5]). However, as the acclaimed pioneers who propounded these ideas and anticipated their significance for the future we must rank the three polymaths, *Joseph-Louis Lagrange* (January 25, 1736–April 10, 1813), *Jean Baptiste Joseph Fourier* (March 21, 1768–May 16, 1830), and *Hermann Minkowski* (June 22, 1864–January 12, 1909).

The translator of the famous elementary textbook of Lagrange [6] Thomas McCormack remarked:

> *In both research and exposition, he totally reversed the methods of his predecessors. They had proceeded in their exposition from special cases by a species of induction; his eye was always directed to the highest and most general points of view; and it was by his suppression of details and neglect of minor, unimportant*

Date: July 10, 2010.

This article bases on a talk at the opening session of the International Conference "Order Analysis and Related Problems of Mathematical Modeling," Vladikavkaz, July 19–24, 2010, dedicated to the 10th anniversary of the Vladikavkaz Scientific Center of the Russian Academy of Sciences.

considerations that he swept the whole field of analysis with a generality of insight and power never excelled, adding to his originality and profundity a conciseness, elegance, and lucidity which have made him the model of mathematical writers.

The pivotal figure was Fourier. Jean-Pierre Kahane wrote in [7, pp. 83–84]:

He himself was neglected for his work on inequalities, what he called "Analyse indéterminée." Darboux considered that he gave the subject an exaggerated importance and did not publish the papers on this question in his edition of the scientific works of Fourier. Had they been published, linear programming and convex analysis would be included in the heritage of Fourier.

David Hilbert lamented the untimely death of Minkowski as follows:[1]

Since my student years Minkowski was my best, most dependable friend who supported me with all the depth and loyalty that was so characteristic of him. Our science, which we loved above all else, brought us together; it seemed to us a garden full of flowers. In it, we enjoyed looking for hidden pathways and discovered many a new perspective that appealed to our sense of beauty, and when one of us showed it to the other and we marvelled over it together, our joy was complete. He was for me a rare gift from heaven and I must be grateful to have possessed that gift for so long. Now death has suddenly torn him from our midst. However, what death cannot take away is his noble image in our hearts and the knowledge that his spirit in us continue to be active.

3. Environment

Assume that X is a real vector space, Y is a *Kantorovich space* also known as a complete vector lattice or a Dedekind complete Riesz space. Let $\mathbb{B} := \mathbb{B}(Y)$ be the *base* of Y, i.e., the complete Boolean algebras of positive projections in Y; and let $m(Y)$ be the universal completion of Y. Denote by $L(X, Y)$ the space of linear operators from X to Y. In case X is furnished with some Y-seminorm on X, by $L^{(m)}(X, Y)$ we mean the *space of dominated operators* from X to Y. As usual, $\{T \le 0\} := \{x \in X \mid Tx \le 0\}$; $\ker(T) = T^{-1}(0)$ for $T : X \longrightarrow Y$. Also, $P \in \mathrm{Sub}(X, Y)$ means that P is *sublinear*, while $P \in \mathrm{PSub}(X, Y)$ means that P is *polyhedral*, i.e., finitely generated. The superscript $^{(m)}$ suggests domination.

4. Kantorovich's Theorem

Find $\mathfrak{X}$ satisfying

$$\begin{tikzcd} X \arrow[r, "A"] \arrow[dr, "B"'] & W \arrow[d, dotted, "\mathfrak{X}"] \\ & Y \end{tikzcd}$$

(1): $(\exists \mathfrak{X})\ \mathfrak{X}A = B \leftrightarrow \ker(A) \subset \ker(B)$.
(2): *If W is ordered by W_+ and $A(X) - W_+ = W_+ - A(X) = W$, then*[2]

$$(\exists \mathfrak{X} \ge 0)\ \mathfrak{X}A = B \leftrightarrow \{A \le 0\} \subset \{B \le 0\}.$$

[1] Cp. [8]. Also see [9].
[2] Cp. [10, p. 51].

5. The Alternative

Let X be a Y-seminormed real vector space, with Y a Kantorovich space. Assume that $A_1, \dots, A_N$ and B belong to $L^{(m)}(X, Y)$.

Then one and only one of the following holds:

(1) *There are $x \in X$ and $b, b' \in \mathbb{B}$ such that $b' \le b$ and*

$$b'Bx > 0, bA_1x \le 0, \dots, bA_Nx \le 0.$$

(2) *There are positive orthomorphisms $\alpha_1, \dots, \alpha_N \in \mathrm{Orth}(m(Y))_+$ such that $B = \sum_{k=1}^N \alpha_k A_k$.*

6. Reals: Hidden Dominance

Lemma 1. *Let X be a vector space over some subfield R of the reals $\mathbb{R}$. Assume that f and g are R-linear functionals on X; in symbols, $f, g \in X^\# := L(X, \mathbb{R})$.*

For the inclusion

$$\{g \le 0\} \supset \{f \le 0\}$$

to hold it is necessary and sufficient that there be $\alpha \in \mathbb{R}_+$ satisfying $g = \alpha f$.

Proof. Sufficiency is obvious.

Necessity: The case of $f = 0$ is trivial. If $f \ne 0$ then there is some $x \in X$ such that $f(x) \in \mathbb{R}$ and $f(x) > 0$. Denote the image $f(X)$ of X under f by R_0. Put $h := g \circ f^{-1}$, i.e. $h \in R_0^\#$ is the only solution for $h \circ f = g$. By hypothesis, h is a positive R-linear functional on R_0. By the Bigard Theorem[3] h can be extended to a positive homomorphism $\bar{h} : \mathbb{R} \longrightarrow \mathbb{R}$, since $R_0 - \mathbb{R}_+ = \mathbb{R}_+ - R_0 = \mathbb{R}$. Each positive automorphism of $\mathbb{R}$ is multiplication by a positive real. As the sought α we may take $\bar{h}(1)$.

The proof of the lemma is complete.

7. Reals: Explicit Dominance

Lemma 2. *Let X be an $\mathbb{R}$-seminormed vector space over some subfield R of $\mathbb{R}$. Assume that $f_1, \dots, f_N$ and g are bounded R-linear functionals on X; in symbols, $f_1, \dots, f_N, g \in X^* := L^{(m)}(X, \mathbb{R})$.*

For the inclusion

$$\{g \le 0\} \supset \bigcap_{k=1}^N \{f_k \le 0\}$$

to hold it is necessary and sufficient that there be $\alpha_1, \dots, \alpha_N \in \mathbb{R}_+$ satisfying $g = \sum_{k=1}^N \alpha_k f_k$.

8. Farkas: Explicit Dominance

Theorem 1. *Assume that $A_1, \dots, A_N$ and B belong to $L^{(m)}(X, Y)$.*

The following are equivalent:

(1) *Given $b \in \mathbb{B}$, the operator inequality $bBx \le 0$ is a consequence of the simultaneous linear operator inequalities $bA_1x \le 0, \dots, bA_Nx \le 0$, i.e.,*

$$\{bB \le 0\} \supset \{bA_1 \le 0\} \cap \dots \cap \{bA_N \le 0\}.$$

[3] Cp. [10, p. 108].

(2) *There are positive orthomorphisms* $\alpha_1, \dots, \alpha_N \in \mathrm{Orth}(m(Y))$ *such that*

$$B = \sum_{k=1}^{N} \alpha_k A_k;$$

i.e., B lies in the operator convex conic hull of $A_1, \dots, A_N$.

9. Boolean Modeling

Cohen's final solution of the problem of the cardinality of the continuum within ZFC gave rise to the Boolean valued models by Scott, Solovay, and Vopěnka.[4] Takeuti coined the term *Boolean valued analysis* for applications of the new models to analysis.[5]

Scott forecasted in 1969:[6]

> *We must ask whether there is any interest in these nonstandard models aside from the independence proof; that is, do they have any mathematical interest? The answer must be yes, but we cannot yet give a really good argument.*

In 2009 Scott wrote:[7]

> *At the time, I was disappointed that no one took up my suggestion. And then I was very surprised much later to see the work of Takeuti and his associates. I think the point is that people have to be trained in Functional Analysis in order to understand these models. I think this is also obvious from your book and its references. Alas, I had no students or collaborators with this kind of background, and so I was not able to generate any progress.*

10. Boolean Valued Universe

Let $\mathbb{B}$ be a complete Boolean algebra. Given an ordinal α, put

$$V_\alpha^{(\mathbb{B})} := \{x \mid (\exists \beta \in \alpha)\ x : \mathrm{dom}(x) \to \mathbb{B} \ \&\ \mathrm{dom}(x) \subset V_\beta^{(\mathbb{B})}\}.$$

The *Boolean valued universe* $\mathbb{V}^{(\mathbb{B})}$ is

$$\mathbb{V}^{(\mathbb{B})} := \bigcup_{\alpha \in \mathrm{On}} V_\alpha^{(\mathbb{B})},$$

with On the class of all ordinals.

The truth value $[\![\varphi]\!] \in \mathbb{B}$ is assigned to each formula φ of ZFC relativized to $\mathbb{V}^{(\mathbb{B})}$.

11. Descending and Ascending

Given φ, a formula of ZFC, and y, a member of $\mathbb{V}^{\mathbb{B}}$; put $A_\varphi := A_{\varphi(\cdot,\ y)} := \{x \mid \varphi(x,\ y)\}$.

The *descent* $A_\varphi{\downarrow}$ of a class A_φ is

$$A_\varphi{\downarrow} := \{t \mid t \in \mathbb{V}^{(\mathbb{B})} \ \&\ [\![\varphi(t,\ y)]\!] = \mathbb{1}\}.$$

If $t \in A_\varphi{\downarrow}$, then it is said that t *satisfies* $\varphi(\cdot,\ y)$ *inside* $\mathbb{V}^{(\mathbb{B})}$.

The *descent* $x{\downarrow}$ of $x \in \mathbb{V}^{(\mathbb{B})}$ is defined as

$$x{\downarrow} := \{t \mid t \in \mathbb{V}^{(\mathbb{B})} \ \&\ [\![t \in x]\!] = \mathbb{1}\},$$

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[4] Cp. [11].

[5] Cp. [12].

[6] Cp. [13].

[7] A letter of April 29, 2009 to S. S. Kutateladze.

i.e. $x{\downarrow} = A_{\cdot \in x}{\downarrow}$. The class $x{\downarrow}$ is a set.

If x is a nonempty set inside $\mathbb{V}^{(\mathbb{B})}$ then

$$(\exists z \in x{\downarrow})[\![(\exists t \in x)\ \varphi(t)]\!] = [\![\varphi(z)]\!].$$

The *ascent* functor acts in the opposite direction.

12. The Reals Within

There is an object $\mathscr{R}$ inside $\mathbb{V}^{(\mathbb{B})}$ modeling $\mathbb{R}$, i.e.,

$$[\![\mathscr{R} \text{ is the reals}]\!] = \mathbb{1}.$$

Let $\mathscr{R}{\downarrow}$ be the descent of the carrier $|\mathscr{R}|$ of the algebraic system $\mathscr{R} := (|\mathscr{R}|, +, \cdot, 0, 1, \leq)$ inside $\mathbb{V}^{(\mathbb{B})}$.

Implement the descent of the structures on $|\mathscr{R}|$ to $\mathscr{R}{\downarrow}$ as follows:

$$x + y = z \leftrightarrow [\![x + y = z]\!] = \mathbb{1};$$
$$xy = z \leftrightarrow [\![xy = z]\!] = \mathbb{1};$$
$$x \leq y \leftrightarrow [\![x \leq y]\!] = \mathbb{1};$$
$$\lambda x = y \leftrightarrow [\![\lambda^{\wedge} x = y]\!] = \mathbb{1}\ (x, y, z \in \mathscr{R}{\downarrow},\ \lambda \in \mathbb{R}).$$

Gordon Theorem.[8] *$\mathscr{R}{\downarrow}$ with the descended structures is a universally complete vector lattice with base $\mathbb{B}(\mathscr{R}{\downarrow})$ isomorphic to $\mathbb{B}$.*

13. Proof of Theorem 1

(2)$\longrightarrow$(1): If $B = \sum_{k=1}^{N} \alpha_k A_k$ for some positive $\alpha_1, \dots, \alpha_N$ in $\mathrm{Orth}(m(Y))$ while $bA_k x \leq 0$ for $b \in \mathbb{B}$ and $x \in X$, then

$$bBx = b\sum_{k=1}^{N} \alpha_k A_k x = \sum_{k=1}^{N} \alpha_k bA_k x \leq 0$$

since orthomorphisms commute and projections are orthomorphisms of $m(Y)$.

(1)$\longrightarrow$(2): Consider the separated Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$ over the base $\mathbb{B}$ of Y. By the Gordon Theorem the ascent $Y{\uparrow}$ of Y is $\mathscr{R}$, the reals inside $\mathbb{V}^{(\mathbb{B})}$.

Using the canonical embedding, we see that $X^{\wedge}$ is an $\mathscr{R}$-seminormed vector space over the standard name $\mathbb{R}^{\wedge}$ of the reals $\mathbb{R}$.

Moreover, $\mathbb{R}^{\wedge}$ is a subfield and sublattice of $\mathscr{R} = Y{\uparrow}$ inside $\mathbb{V}^{(\mathbb{B})}$.

Put $f_k := A_k{\uparrow}$ for all $k := 1, \dots, N$ and $g := B{\uparrow}$. Clearly, all $f_1, \dots, f_N, g$ belong to $(X^{\wedge})^*$ inside $\mathbb{V}^{\mathbb{B}}$.

Define the finite sequence

$$f : \{1, \dots, N\}^{\wedge} \longrightarrow (X^{\wedge})^*$$

as the ascent of $(f_1, \dots, f_N)$. In other words, the truth values are as follows:

$$[\![f_{k^{\wedge}}(x^{\wedge}) = A_k x]\!] = \mathbb{1}, \quad [\![g(x^{\wedge}) = Bx]\!] = \mathbb{1}$$

for all $x \in X$ and $k := 1, \dots, N$.

Put

$$b := [\![A_1 x \leq 0^{\wedge}]\!] \wedge \dots \wedge [\![A_N x \leq 0^{\wedge}]\!].$$

Then $bA_k x \leq 0$ for all $k := 1, \dots, N$ and $bBx \leq 0$ by (1).

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[8]Cp. [11, p. 349].

Therefore,

$$[\![A_1x \leq 0^\wedge]\!] \wedge \dots \wedge [\![A_Nx \leq 0^\wedge]\!] \leq [\![Bx \leq 0^\wedge]\!].$$

In other words,

$$[\![(\forall k := 1^\wedge, \dots, N^\wedge) f_k(x^\wedge) \leq 0^\wedge]\!]$$

$$= \bigwedge_{k:=1,\dots,N} [\![f_{k^\wedge}(x^\wedge) \leq 0^\wedge]\!] \leq [\![g(x^\wedge) \leq 0^\wedge]\!].$$

By Lemma 2 inside $\mathbb{V}^{(\mathbb{B})}$ and the maximum principle of Boolean valued analysis, there is a finite sequence $\alpha : \{1^\wedge, \dots, N^\wedge\} \longrightarrow \mathscr{R}_+$ inside $\mathbb{V}^{(\mathbb{B})}$ satisfying

$$[\![(\forall x \in X^\wedge)\ g(x) = \sum_{k=1^\wedge}^{N^\wedge} \alpha(k) f_k(x)]\!] = \mathbb{1}.$$

Put $\alpha_k := \alpha(k^\wedge) \in \mathscr{R}_+{\downarrow}$ for $k := 1, \dots, N$. Multiplication by an element in $\mathscr{R}{\downarrow}$ is an orthomorphism of $m(Y)$. Moreover,

$$B = \sum_{k=1}^{N} \alpha_k A_k,$$

which completes the proof.

14. Counterexample: No Dominance

Lemma 1, describing the consequences of a single inequality, does not restrict the class of functionals under consideration.

The analogous version of the Farkas Lemma simply fails for two simultaneous inequalities in general.

The inclusion $\{f = 0\} \subset \{g \leq 0\}$ equivalent to the inclusion $\{f = 0\} \subset \{g = 0\}$ does not imply that f and g are proportional in the case of an arbitrary subfield of $\mathbb{R}$. It suffices to look at $\mathbb{R}$ over the rationals $\mathbb{Q}$, take some discontinuous $\mathbb{Q}$-linear functional on $\mathbb{Q}$ and the identity automorphism of $\mathbb{Q}$.

15. Reconstruction: No Dominance

Theorem 2. *Take A and B in $L(X, Y)$. The following are equivalent:*
(1) $(\exists \alpha \in \mathrm{Orth}(m(Y)))\ B = \alpha A$;
(2) *There is a projection $\varkappa \in \mathbb{B}$ such that*

$$\{\varkappa bB \leq 0\} \supset \{\varkappa bA \leq 0\}; \quad \{\neg\varkappa bB \leq 0\} \supset \{\neg\varkappa bA \geq 0\}$$

for all $b \in \mathbb{B}$.[9]

Proof. Boolean valued analysis reduces the claim to the scalar case. Applying Lemma 1 twice and writing down the truth values, complete the proof.

[9] As usual, $\neg\varkappa := \mathbb{1} - \varkappa$.

16. Interval Operators

Let X be a vector lattice. An *interval operator* $\mathbf{T}$ from X to Y is an order interval $[\underline{T}, \overline{T}]$ in $L^{(r)}(X, Y)$, with $\underline{T} \leq \overline{T}$.[10] The interval equation $\mathbf{B} = \mathfrak{X}\mathbf{A}$ has a *weak interval solution* provided that $(\exists \mathfrak{X})(\exists A \in \mathbf{A})(\exists \mathbf{B} \in \mathbf{B})\ \mathbf{B} = \mathfrak{X}\mathbf{A}$.

Given an interval operator $\mathbf{T}$ and $x \in X$, put

$$P_{\mathbf{T}}(x) = \overline{T}x_+ - \underline{T}x_-.$$

Call $\mathbf{T}$ *adapted* in case $\overline{T} - \underline{T}$ is the sum of finitely many disjoint addends.

Put $\sim (x) := -x$ for all $x \in X$.

17. Interval Equations

Theorem 3. *Let X be a vector lattice, and let Y be a Kantorovich space. Assume that $\mathbf{A}_1, \dots, \mathbf{A}_N$ are adapted interval operators and $\mathbf{B}$ is an arbitrary interval operator in the space of order bounded operators $L^{(r)}(X, Y)$.*

The following are equivalent:

(1) *The interval equation*

$$\mathbf{B} = \sum_{k=1}^{N} \alpha_k \mathbf{A}_k$$

has a weak interval solution $\alpha_1, \dots, \alpha_N \in \mathrm{Orth}(Y)_+$.

(2) *For all $b \in \mathbb{B}$ we have*

$$\{b\mathfrak{B} \geq 0\} \supset \{b\mathfrak{A}_1^{\sim} \leq 0\} \cap \dots \cap \{b\mathfrak{A}_N^{\sim} \leq 0\},$$

where $\mathfrak{A}_k^{\sim} := P_{\mathbf{A}_k} \circ \sim$ for $k := 1, \dots, N$ and $\mathfrak{B} := P_{\mathbf{B}}$.

18. Inhomogeneous Inequalities

Theorem 4. *Let X be a Y-seminormed real vector space, with Y a Kantorovich space. Assume given some dominated operators $A_1, \dots, A_N, B \in L^{(m)}(X, Y)$ and elements $u_1, \dots, u_N, v \in Y$. The following are equivalent:*

(1) *For all $b \in \mathbb{B}$ the inhomogeneous operator inequality $bBx \leq bv$ is a consequence of the consistent simultaneous inhomogeneous operator inequalities $bA_1x \leq bu_1, \dots, bA_Nx \leq bu_N$, i.e.,*

$$\{bB \leq bv\} \supset \{bA_1 \leq bu_1\} \cap \dots \cap \{bA_N \leq bu_N\}.$$

(2) *There are positive orthomorphisms $\alpha_1, \dots, \alpha_N \in \mathrm{Orth}(m(Y))$ satisfying*

$$B = \sum_{k=1}^{N} \alpha_k A_k; \quad v \geq \sum_{k=1}^{N} \alpha_k u_k.$$

[10] Cp. [15].

19. Inhomogeneous Matrix Inequalities

Theorem 5.[11] *Let X be a Y-seminormed real vector space, with Y a Kantorovich space. Assume that $A \in L^{(m)}(X, Y^s)$, $B \in L^{(m)}(X, Y^t)$, $u \in Y^s$, and $v \in Y^t$, where s and t are some naturals.*

The following are equivalent:

(1) For all $b \in \mathbb{B}$ the inhomogeneous operator inequality $bBx \le bv$ is a consequence of the consistent inhomogeneous inequality $bAx \le bu$, i.e., $\{bB \le bv\} \supset \{bA \le bu\}$.

(2) There is some $s \times t$ matrix with entries positive orthomorphisms of $m(Y)$ such that $B = \mathfrak{X}A$ and $\mathfrak{X}u \le v$ for the corresponding linear operator $\mathfrak{X} \in L_+(Y^s, Y^t)$.

20. Complex Scalars

Theorem 6. *Let X be a Y-seminormed complex vector space, with Y a Kantorovich space. Assume given some $u_1, \dots, u_N, v \in Y$ and dominated operators $A_1, \dots, A_N, B \in L^{(m)}(X, Y_{\mathbb{C}})$ from X into the complexification $Y_{\mathbb{C}} := Y \otimes iY$ of Y.*[12] *Assume further that the simultaneous inhomogeneous inequalities $|A_1x| \le u_1, \dots, |A_Nx| \le u_N$ are consistent. Then the following are equivalent:*

(1) $\{b|B(\cdot)| \le bv\} \supset \{b|A_1(\cdot)| \le bu_1\} \cap \dots \cap \{b|A_N(\cdot)| \le bu_N\}$ for all $b \in \mathbb{B}$.

(2) There are complex orthomorphisms $c_1, \dots, c_N \in \mathrm{Orth}(m(Y)_{\mathbb{C}})$ satisfying

$$B = \sum_{k=1}^{N} c_k A_k; \quad v \ge \sum_{k=1}^{N} |c_k| u_k.$$

21. Inhomogeneous Sublinear Inequalities

Lemma 3. *Let X be a real vector space. Assume that $p_1, \dots, p_N \in \mathrm{PSub}(X) := \mathrm{PSub}(X, \mathbb{R})$ and $p \in \mathrm{Sub}(X)$. Assume further that $v, u_1, \dots, u_N \in \mathbb{R}$ make consistent the simultaneous sublinear inequalities $p_k(x) \le u_k$, with $k := 1, \dots, N$.*

The following are equivalent:

(1) $\{p \ge v\} \supset \bigcap\limits_{k=1}^{N} \{p_k \le u_k\}$;

(2) There are $\alpha_1, \dots, \alpha_N \in \mathbb{R}_+$ satisfying

$$(\forall x \in X)\ p(x) + \sum_{k=1}^{N} \alpha_k p_k(x) \ge 0, \quad \sum_{k=1}^{N} \alpha_k u_k \le -v.$$

Proof. (2) $\longrightarrow$ (1): If x is a solution to the simultaneous inhomogeneous inequalities $p_k(x) \le u_k$ with $k := 1, \dots, N$, then

$$0 \le p(x) + \sum_{k=1}^{N} \alpha_k p_k(x) \le p(x) + \sum_{k=1}^{N} \alpha_k u_k(x) \le p(x) - v.$$

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[11] Cp. [14].

[12] Cp. [3, p. 338].

(1) $\longrightarrow$ (2): Given $(x,t) \in X \times \mathbb{R}$, put $\bar{p}_k(x,t) := p_k(x) - tu_k$, $\bar{p}(x,t) := p(x) - tv$ and $\tau(x,t) := -t$. Clearly, $\tau, \bar{p}_1, \ldots, \bar{p}_N \in \mathrm{PSub}(X \times \mathbb{R})$ and $\bar{p} \in \mathrm{Sub}(X \times \mathbb{R})$. Take

$$(x,t) \in \{\tau \leq 0\} \cap \bigcap_{k=1}^{N} \{\bar{p}_k \leq 0\}.$$

If, moreover, $t > 0$; then $u_k \geq p_k(x/t)$ for $k := 1, \ldots, N$ and so $p(x/t) \leq v$ by hypothesis. In other words $(x,t) \in \{\bar{p} \leq 0\}$. If $t = 0$ then take some solution $\bar{x}$ of the simultaneous inhomogeneous polyhedral inequalities under study.

Since $x \in K := \bigcap_{k=1}^{N} \{p_k \leq 0\}$; therefore, $p_k(\bar{x} + x) \leq p(x) + p_k(x) \leq u_k$ for all $k := 1, \ldots, N$. Hence, $p(\bar{x} + x) \geq v$ by hypothesis. So the sublinear functional p is bounded below on the cone K. Consequently, p assumes only positive values on K. In other words, $(x,0) \in \{\bar{p} \leq 0\}$. Thus

$$\{\bar{p} \geq 0\} \supset \bigcap_{k=1}^{N} \{\bar{p}_k \leq 0\}$$

and by Lemma 2.2. of [1] there are positive reals $\alpha_1, \ldots, \alpha_N, \beta$ such that for all $(x,t) \in X \times \mathbb{R}$ we have

$$\bar{g}(x) + \beta\tau(x) + \sum_{k=1}^{N} \alpha_k \bar{p}_k(x) \geq 0.$$

Clearly, the so-obtained parameters $\alpha_1, \ldots, \alpha_N$ are what we sought for. The proof of the lemma is complete.

Corollary. *Let X be an $\mathbb{R}$-seminormed complex vector space. Given are $u_1, \ldots, u_N, v \in Y$ and bounded complex linear functionals $f_1, \ldots, f_N, f \in X^*$. Assume that consistent are the simultaneous inhomogeneous inequalities $|f_1(x)| \leq u_1, \ldots, |f_N(x)| \leq u_N$. Then the following are equivalent:*

(1) *The inequality $|g(x)| \leq v$ is a consequence of the simultaneous inequalities $|f_1(x)| \leq u_1, \ldots, |f_N(x)| \leq u_N$, i.e.*

$$\{|g(\cdot)| \leq v\} \supset \{|f_1(\cdot)| \leq u_1\} \cup \cdots \cup \{|f_N(\cdot)| \leq u_N\};$$

(2) *There are $c_1, \ldots, c_N \in \mathbb{C}$ such that*

$$g = \sum_{k=1}^{N} c_k f_k, \quad v \geq \sum_{k=1}^{N} |c_k| u_k.$$

Proof. (2) $\longrightarrow$ (1): If $x \in \bigcap_{k=1}^{N} \{|f_k(\cdot)| \leq u_k\}$ then

$$|g(x)| = \Big|\sum_{k=1}^{N} c_k f_k(x)\Big| \leq \sum_{k=1}^{N} |c_k f_k(x)| \leq \sum_{k=1}^{N} |c_k| u_k \leq v.$$

(1) $\longrightarrow$ (2): Consider the realification $X_{\mathbb{R}}$ of X and the sublinear functionals $p(x) := -\operatorname{Re} g(x)$ and $p_k(x) := |f_k(x)|$, where $k := 1, \ldots, N$ and $x \in X_{\mathbb{R}}$. Clearly, Lemma 3 applies and

$$\{p \geq -v\} \supset \bigcap_{k=1}^{N} \{p_k \leq u_k\}.$$

Hence, there are positive reals $\alpha_1, \dots, \alpha_N$ satisfying

$$(\forall x \in X_{\mathbb{R}}) - \operatorname{Re} g(x) + \sum_{k=1}^{N} \alpha_k |f_k(x)| \geq 0; \quad \sum_{k=1}^{N} \alpha_k u_k \leq v.$$

By subdifferential calculus there are complexes $\theta_k, |\theta_k| = 1, k := 1, \dots, N$, such that $g = \sum \alpha_k \theta_k f_k$. Put $c_k := \alpha_k \theta_k$. Obviously, $\sum_{k=1}^{N} |c_k| u_k = \sum_{k=1}^{N} \alpha_k |\theta_k| u_k \leq v$. The proof of the corollary is complete.

Remark. Theorem 6 (which is Theorem 3.1 in [20] supplied with a slightly dubious proof) is a Boolean valued interpretation of the Corollary.

Theorem 7. *Let X be a Y-seminormed real vector space, with Y a Kantorovich space. Given are some dominated polyhedral sublinear operators $P_1, \dots, P_N \in \mathrm{PSub}^{(m)}(X, Y)$ and a dominated sublinear operator $P \in \mathrm{Sub}^{(m)}(X, Y)$. Assume further that $u_1, \dots, u_N, v \in Y$ make consistent the simultaneous inhomogeneous inequalities $P_1(x) \leq u_1, \dots, P_N(x) \leq u_N, P(x) \geq v$.*

The following are equivalent:

(1) *For all $b \in \mathbb{B}$ the inhomogeneous sublinear operator inequality $bP(x) \geq v$ is a consequence of the simultaneous inhomogeneous sublinear operator inequalities $bP_1(x) \leq u_1, \dots, bP_N(x) \leq u_N$, i.e.,*

$$\{bP \geq v\} \supset \{bP_1 \leq u_1\} \cap \dots \cap \{bP_N \leq u_N\};$$

(2) *There are positive $\alpha_1, \dots, \alpha_N \in \mathrm{Orth}(m(Y))$ satisfying*

$$(\forall x \in X)\ P(x) + \sum_{k=1}^{N} \alpha_k P_k(x) \geq 0, \quad \sum_{k=1}^{N} \alpha_k u_k \leq -v.$$

22. Lagrange's Principle

The finite value of the constrained problem

$$P_1(x) \leq u_1, \dots, P_N(x) \leq u_N, \quad P(x) \longrightarrow \inf$$

is the value of the unconstrained problem for an appropriate Lagrangian without any constraint qualification other that polyhedrality.

The Slater condition allows us to eliminate polyhedrality as well as considering a unique target space. This is available in a practically unrestricted generality.[13]

About the new trends relevant to the Farkas Lemma see [16]–[20].

23. Freedom and Inequality

Convexity is the theory of linear inequalities in disguise.

Abstraction is the freedom of generalization. Freedom is the loftiest ideal and idea of man, but it is demanding, limited, and vexing. So is abstraction. So are its instances in convexity, hence, in simultaneous inequalities.

We definitely feel truth, but we cannot define truth properly. That is what Alfred Tarski explained to us in the 1930s.[14] We pursue truth by way of proof, as wittily phrased by Saunders Mac Lane.[15] Mathematics becomes logic.

The freedom of set theory empowered us with the Boolean valued models yielding a lot of surprising and unforeseen visualizations of the ingredients of mathematics.

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[13] Cp. [10].
[14] Cp. [21].
[15] Cp. [22].

Many promising opportunities are open to modeling the powerful habits of reasoning and verification.

Logic organizes and orders our ways of thinking, manumitting us from conservatism in choosing the objects and methods of research. Logic of today is a fine instrument and institution of mathematical freedom. Logic liberates mathematics by model theory.

Model theory evaluates and counts truth and proof. The chase of truth not only leads us close to the truth we pursue but also enables us to nearly catch up with many other instances of truth which we were not aware nor even foresaw at the start of the rally pursuit. That is what we have learned from Boolean valued analysis.

Freedom presumes liberty and equality. Inequality paves way to freedom.

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